,

ON NATURAL DEFORMATIONS OF SYMPLECTIC AUTOMORPHISMS OF MANIFOLDS OF $K3^{[n]}$ type

GIOVANNI MONGARDI

Abstract. In the present paper we prove that finite symplectic groups of automorphisms of manifolds of $K3^{[n]}$ type can be obtained by deforming natural morphisms arising from $K3$ surfaces if and only if they satisfy a certain numerical condition.

Keywords: Symplectic Automorphisms, $K3^{[n]}$ type, Natural morphism
MSC 2010 classification: 14J50

1. Introduction

The present paper is devoted to a natural question concerning deformations of automorphisms of hyperkähler manifolds. Roughly speaking, given a $K3$ surface S the group $\mathrm{Aut}(S)$ induces automorphisms of the Hilbert scheme $S^{[n]}$ of n points of S. These automorphisms are called natural. Let X be a hyperkähler manifold deformation equivalent to some $S^{[n]}$ and let G be a group of automorphisms of X. One can ask whether it is possible to deform X together with G to some $(S^{[n]}, G)$, where G is a group of natural automorphisms. In the following we give a positive answer for all finite symplectic automorphism groups whose action on $H^2(X)$ is the natural one and for several different dimensions (cf. Theorem 2.5). We remark that having the natural action on $H^2(X)$ is a necessary condition, since this action is constant under smooth deformations.

There have been several works concerning automorphisms of $K3$ surfaces, we will refer to the foundational work of Nikulin [11], later improved by Mukai [10] in the nonabelian case. By the work of Mukai [10] there are 79 possible finite groups of symplectic automorphisms on $K3$ surfaces and, by a recent classification due to Hashimoto [5], there are 84 different possibilities for their action on H^2. Our result holds for all these 84 cases as long as the hypothesis of the global Torelli theorem are satisfied.

In the case of manifolds of $K3^{[n]}$ type the notion of natural morphisms was introduced by Boissière [3] and further analyzed by him and Sarti [4]. In the particular case of symplectic involutions on manifolds of $K3^{[2]}$ type our result is proven in [9].

Notations. If L is a lattice and $G \subset O(L)$ we denote by $T_G(L) := L^G$ the invariant sublattice and by $S_G(L) := T_G(L)^\perp$ the coinvariant sublattice. For $G \subset \mathrm{Aut}(X)$ and $H^2(X, \mathbb{Z})$ endowed with a quadratic form, we denote $T_G(X) := T_G(H^2(X, \mathbb{Z}))$ the invariant sublattice and $S_G(X) := S_G(H^2(X, \mathbb{Z}))$ the coinvariant sublattice. Let X be a hyperkähler manifold and let $G \subset \mathrm{Aut}(X)$. The group G is called symplectic if it acts trivially on $H^{2,0}(X)$, i. e. it preserves the symplectic form. We denote by $\mathrm{Aut}_s(X)$ the subgroup of automorphisms of X preserving the symplectic form. We will call manifolds of $K3^{[n]}$ type all manifold deformation equivalent to the Hilbert scheme of n points on a K3 surface.

Preliminaries. In this section we gather some useful results for ease of reference. The reader interested in hyperkähler manifolds can consult [7] and [8] for further references and for a broader treatment of the subject.

A hyperkähler manifold is a simply connected compact Kähler manifold whose $H^{2,0}$ is generated by a symplectic form.

Theorem 1.1. *Let X be a hyperkähler manifold of dimension $2n$. Then there exists a canonically defined pairing $(\ ,\ )_X$ on $H^2(X,\mathbb{C})$, the Beauville-Bogomolov pairing, which is a deformation and birational invariant. This form makes $H^2(X,\mathbb{Z})$ a lattice of signature $(3, b_2(X)-3)$.*

For every hyperkähler manifold X and every Kähler class ω there exists a family of smooth deformations of X over the base $\mathbb{P}^1$. This family is called *twistor family* and denoted $TW_\omega(X)$.

Example 1. *Let X be a hyperkähler manifold of $K3^{[n]}$ type. Then $H^2(X,\mathbb{Z})$ endowed with its Beauville-Bogomolov pairing is isomorphic to the lattice*

$$L_n := H^2(K3,\mathbb{Z}) \oplus (2-2n). \tag{1}$$

If X is hyperkähler we call a marking of X any isometry between $H^2(X,\mathbb{Z})$ and a lattice M. There exists a moduli space of marked hyperkähler manifolds with $H^2(X,\mathbb{Z}) \cong M$ and we denote it by $\mathcal{M}_M$.
We will often consider the induced action of $\mathrm{Aut}(X)$ on $O(H^2(X,\mathbb{Z}))$ for a manifold X of $K3^{[n]}$ type. For a general hyperkähler manifold this map might not be injective but in our case it is:

Lemma 1.2. *Let X be a manifold of $K3^{[n]}$ type. Then the map*

$$\nu(X) : \mathrm{Aut}(X) \to O(H^2(X,\mathbb{Z})) \tag{2}$$

is injective.

Proof. By [6, Theorem 2.1] the kernel of $\nu(X)$ is invariant under smooth deformations. Beauville [1, Lemma 3] proved that, if S is a $K3$ surface with no non-trivial automorphisms, then $\mathrm{Aut}(S^{[n]}) = \{Id\}$, therefore $\{Id\} = Ker(\nu(S^{[n]})) = Ker(\nu(X))$. □

The following is a very important theorem which is essential in the proof of our main result. The only truly restrictive hypothesis of Theorem 2.5 is one of the hypotheses of the following:

Theorem 1.3 (Global Torelli, Verbitsky, Markman and Huybrechts)**.** *Let X and Y be two hyperkähler manifolds of $K3^{[n]}$ type and let $n-1$ be a prime power. Suppose $\psi : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ is an isometry preserving the Hodge structure. Then there exists a birational map $\phi : X \dashrightarrow Y$.*

Let M be a lattice of signature $(3,r)$. We define $\Omega_M = \mathbb{P}(\{x \in M\otimes\mathbb{C} \mid x^2 = 0, (x,\overline{x}) > 0\})$ as the period domain for the lattice M. It is an open subset of a quadric hypersurface inside $\mathbb{P}(M\otimes\mathbb{C})$.

In the particular case where $M \cong H^2(X,\mathbb{Z})$ for some hyperkähler manifold X, there exists a natural map, the period map $\mathcal{P}$, between the moduli space $\mathcal{M}_M$ and the period domain Ω_M.

Moreover, when Theorem 1.3 holds, two marked manifolds having the same period are birational.

The images of twistor families in $\mathcal{M}_M$ through the period map are called *twistor lines*. A fundamental property of period domains is that they are connected by twistor lines (see [8, Proposition 3.7] or [2]).

2. Deformations of pairs

Definition 2.1. *Let X be a manifold and let $G \subset \mathrm{Aut}(X)$. A G deformation of X (or a deformation of the pair (X, G)) consists of the following data:*

- *A flat family $\mathcal{X} \to B$, B connected and $\mathcal{X}$ smooth, and a distinguished point $0 \in B$ such that $\mathcal{X}_0 \cong X$.*
- *A faithful action of the group G on $\mathcal{X}$ inducing fibrewise faithful actions of G.*

Two pairs (X, G) and (Y, H) are deformation equivalent if (Y, H) lies in a G deformation of X.

The first interesting remark is that, to some extent, all symplectic automorphism groups of a hyperkähler manifold can be deformed:

Remark 1. *Let X be a hyperkähler manifold such that $G \subset \mathrm{Aut}_s(X)$ and $|G| < \infty$. Let ω be a G invariant Kähler class. Then $TW_\omega(X)$ is a G deformation of X over $\mathbb{P}^1$.*

There is also a notion of local universal G deformation, for a proof of its existence we refer to [9].

Lemma 2.2. *Let X be a manifold of $K3^{[n]}$ type and let $G \subset \mathrm{Aut}_s(X)$. Then there exists a universal local G deformation of X sitting inside $Def(X)$. It is locally given by the G-invariant part of $H^1(T_X)$ and it is of dimension $\mathrm{rank}(T_G(X)) - 2$. Moreover two birational manifolds with isomorphic actions of G on cohomology have intersecting local G-deformations.*

Proof. Let X be birational to Y and let the action of G on $H^2(X)$ coincide with the action of G on $H^2(Y)$ induced by the birational transformation between X and Y. Let us take a representative U of $Def(X)$ and let x be a very general point inside U^G, which is a representative of the local G deformations of X and Y. Let $\mathcal{Y}_x$ and $\mathcal{X}_x$ be the two hyperkähler manifolds corresponding to x on U^G. We have $Pic(\mathcal{Y}_x) = Pic(\mathcal{X}_x) = S_G(X)$ and $\mathcal{Y}_x$ is birational to $\mathcal{X}_x$. However any G invariant Kähler class on $\mathcal{Y}_x$ is orthogonal to $Pic(\mathcal{Y}_x)$ and therefore also to the set of effective curves on $\mathcal{Y}_x$, which is therefore empty. Thus the Kähler cone of $\mathcal{Y}_x$ coincides with the positive cone and $\mathcal{Y}_x = \mathcal{X}_x$. □

We remark that the local G deformations around two birational manifolds might not meet for a nonsymplectic group G.

Definition 2.3. *Let S be a $K3$ surface and let $G \subset \mathrm{Aut}_s(S)$ be a group of symplectic automorphisms on S. G induces a group of symplectic morphisms on $S^{[n]}$ which we still denote as G. We call the pair $(S^{[n]}, G)$ a natural pair, following* [3]. *We call standard any pair (X, H) deformation equivalent to a natural pair.*

A natural question is asking under which condition a pair (X, G) is standard. In the rest of the paper we make the following assumption and we prove that it is equivalent to (X, G) being standard.

Definition 2.4. *Let X be a manifold of $K3^{[n]}$ type and let $G \subset \mathrm{Aut}_s(X)$. The group G is* numerically standard *if the following holds*

- $S_G(X) \cong S_H(S)$,
- $T_G(X) \cong T_H(S) \oplus \langle t \rangle$.
- $t^2 = -2(n-1)$, $(t, H^2(X,\mathbb{Z})) = 2(n-1)\mathbb{Z}$.

For some $K3$ surface S and some $H \subset \mathrm{Aut}_s(S)$ such that $H \cong G$.

Notice that for a standard pair (X,G) the group G is numerically standard, since by [4] a natural pair is numerically standard. Now the main result of the paper can be explicitly stated:

Theorem 2.5. *Let X be a manifold of $K3^{[n]}$ type and let $n-1$ be a prime power. Let $G \subset \mathrm{Aut}_s(X)$ be a finite group of numerically standard automorphisms. Then (X,G) is a standard pair.*

In this section we prove Theorem 2.5 using some properties of a particular period domain defined by the action of a finite group G of symplectic automorphisms of a manifold X of $K3^{[n]}$ type.

Definition 2.6. *Let M be a lattice of signature $(3,r)$ and let $G \subset O(M)$. We call $\Omega_{G,M}$ the set of points ω in the period domain Ω_M such that $\omega \in T_G(M) \otimes \mathbb{C}$.*

Definition 2.7. *Let $\mathcal{M}_n := \mathcal{M}_{L_n}$ be the moduli space of marked manifolds of $K3^{[n]}$ type and let $G \subset \mathrm{Aut}_s(X)$ for some marked $(X,f) \in \mathcal{M}_n$. Let us denote with G the group of isometries induced by G on the lattice L_n and let $\Omega_{G,n} := \Omega_{G,L_n}$ be as above. Then we define $\mathcal{M}_{G,n} \subset \mathcal{M}_n$ as the counterimage through the period map of $\Omega_{G,n}$.*

By the following remark the set $\mathcal{M}_{G,n}$ is the set of marked pairs (X,f) such that $f^{-1}(S_G(L_n)) \subset Pic(X)$ for an appropriate marking f and $\Omega_{G,n}$ is just the period domain $\Omega_{T_G(L_n)}$.

Remark 2. *Let X be a hyperkähler manifold and let $G \subset \mathrm{Aut}_s(X)$ be a finite group. Then $T_G(X)$ contains $T(X)$ and $S_G(X) \subset Pic(X)$. Moreover $T_G(X)$ has signature $(3,r)$ for some $r \geq 0$. A proof of this fact can be found in* [1, Proposition 6].

This means that, through a chain of twistor families, we can connect any marked point $(X,f) \in \mathcal{M}_{G,n}$ with $G \subset \mathrm{Aut}_s(X)$ numerically standard to a marked point (Y,g) that has the same period of a natural pair $(S^{[n]},G)$ for an appropriate marking f' of $S^{[n]}$. Since by Remark 1 twistor families are G deformations, we have that (X,G) and (Y,G) are deformation equivalent.

Proof of Theorem 2.5. *Let X be a manifold of $K3^{[n]}$ type and let $n-1$ be a prime power. Let $G \subset \mathrm{Aut}_s(X)$ be a finite numerically standard group of symplectic automorphisms. Since $\Omega_{G,n}$ is connected by twistor lines, (X,G) is deformation equivalent to (Y,G) and $\mathcal{P}(Y,f) = \mathcal{P}(S^{[n]},f') \in \Omega_{G,n}$. Here S is a $K3$ surface with $G \subset \mathrm{Aut}_s(S)$ and $Pic(S) = S_G(S)$, i. e. the very general $K3$ surface with $G \subset Aut_s(S)$. By Theorem 1.3 there is a birational map ϕ between Y and $S^{[n]}$ which gives an induced action of G on $S^{[n]}$ (possibly nonregular). Let us denote by H the group induced on $S^{[n]}$ by ϕ and let us keep calling G the group induced by the automorphisms of S. We obtain our claim by proving that $H = G$ (as actions on $S^{[n]}$), since in that case (Y,G) and $(S^{[n]},H)$ would be deformation equivalent through their local universal G-deformations.*

Notice that, by the assumption on the numerical standardness, the actions of G *and* H *already coincide on* $H^2(S^{[n]},\mathbb{Z})$. *Let now* $g \in G$ *and let* h *be the element of* H *such that* $g^* = h^*$ *in* $H^2(S^{[n]},\mathbb{Z})$. *Let* r *be the order of* g. *Then* $g \circ h^{r-1}$ *induces the identity on* $H^2(S^{[n]},\mathbb{Z})$. *Therefore, by Lemma 1.2,* $g^{-1} = h^{r-1}$, *which implies* $G = H$ *as group of automorphisms of* $S^{[n]}$.

Acknowledgements

The present paper is an improvement of a result contained in my PhD thesis. I am very grateful to my former advisor, Prof. K. G. O'Grady for his support and to Prof. D. Huybrechts for his advice.
Partially supported by the Research Network Program GDRE-GRIFGA.

References

[1] A. Beauville, Some remarks on Kähler manifolds with $c_1 = 0$: Prog. Math. vol. 39 (1983), 1–26
[2] A. Beauville et al., Géométrie des surfaces K3: modules et Périodes, Astérisque vol. 126 (1985), 1–193
[3] S. Boissière, Automorphismes naturels de l'espace de Douady de points sur une surface: Canad. J. Math. vol. 64 no. 1 (2012), 3–23
[4] S. Boissière and A. Sarti, A note on automorphisms and birational transformations of holomorphic symplectic manifolds: Proc. Amer. Math. Soc. vol. 140 no. 12 (2012) 4053–4062
[5] K. Hashimoto, Finite symplectic actions on the $K3$ lattice : Nagoya Math. J. vol. 206 (2012), 99–153
[6] B. Hassett and Y. Tschinkel, Hodge theory and Lagrangian planes on generalized Kummer fourfolds: Preprint arXiv:1004.0046v1 [Math.AG]
[7] D. Huybrechts, Compact Hyperkähler manifolds, Calabi-Yau manifolds and related geometries (Nordfjordeid, 2001) Universitext
[8] D. Huybrechts, A global Torelli theorem for Hyperkähler manifolds (after Verbitsky), Semin. Bourbaki n. 1040 (2010-2011)
[9] G. Mongardi, Symplectic involutions on deformations of $K3^{[2]}$, Centr. Eur. J. Math. vol. 10 no. 4 (2012) 1472–1485
[10] S. Mukai, Finite groups of automorphisms of $K3$ surfaces and the Mathieu group, Invent. Math. vol. 94 (1988), 183–221
[11] V.V. Nikulin, Finite automorphism groups of Kahlerian K3 surfaces (Russian): Trudy Moscov Math. Obshch. vol. 38 (1979), 75–137

Mathematisches Institut der Universität Bonn, Endenicher Allee, 60, 53115 Bonn, Germany